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Comments on "Heat Conduction in a Bounded, Anisotropic Medium"

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REFERENCE 1 is one of a series of very interesting papers that have been published in recent years on the development of the so-called "William's Method" (velocity or acceleration methods) for the solution of eigenvalue problems of a variety of field theories. The method is characterized by its acceptance of time-dependent boundary conditions and by its improved convergence properties relative to a conventional modal approach. Three remarks are appropriate.

1) Soon after the appearance of the Mindlin-Goodman technique[2] for the solution of eigenvalue problems with time-dependent boundary conditions, it became apparent[3] that responses expressed according to this approach can take the form of the sum of the quasi-static solution and a product series involving eigenfunctions. Thus, as a comparison, for example, of Ref. 3 of this Comment with Ref. 2 of the Technical Note (Ref. 1) will verify, the Mindlin-Goodman technique is precisely equivalent to William's Method. There is a very substantial literature on this type of Mindlin-Goodman solution which has appeared in the last decade (e.g., Refs. 3, 4, 5, and 6). This work, which encompasses very general forms of field theories including elasticity and heat conduction, parallels in content and, in most cases, precedes the developments of William's Method. It can be concluded that those wishing to employ William's Method can usually extract the desired information from the appropriate Mindlin-Goodman type solution development.

2) Conventional modal theory for the classical field theories can accept time-dependent boundary conditions (e.g., Refs. 7, 8, 9, and 10), although convergence questions can arise.

3) These conventional modal theories can be transformed into a quasi-static solution plus the eigenfunction product series (i.e., William's solution). Vibration texts often use integration by parts[11] to accomplish this transformation.

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Comment on "An Automatic Node-Relabeling Scheme for Bandwidth Minimization of Stiffness Matrices"

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THE recent Note by Akyuz and Utku[1] is significant in that very little published work is available on this topic. In the case where coupling between elements is simple, the method is efficient and leads to a true minimum. However, when the coupling is more complicated, interchanges of single pairs of variables do not insure monotonic reduction of the bandwidth to a minimum. In general, the bandwidth is reduced but not necessarily to the minimum.

Suppose the current sequence of the variables is sequence A and a minimum bandwidth is given by sequence B; then all possible routes from A to B, using single interchanges, may pass through sequences that give a greater bandwidth than does sequence A. In this case the algorithm fails. For example, the plate assembly labeled in the sequence shown in Fig. 1a, and with a single variable at each node, has a stiffness matrix image of bandwidth 11 (see Fig. 1b). Using the technique of Ref. 1, the only possible single interchanges that do not increase the bandwidth in the first sweep are 3 with 4 and 9 with 10. These changes do not reduce the bandwidth and the program terminates after the next sweep.

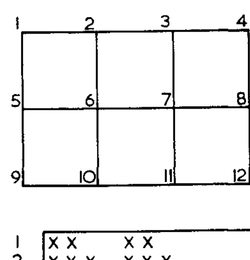

Fig. 1a Plate assembly.

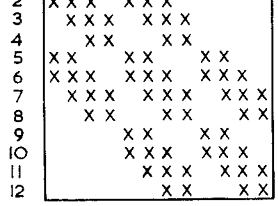

Fig. 1b Matrix image for the plate assembly.

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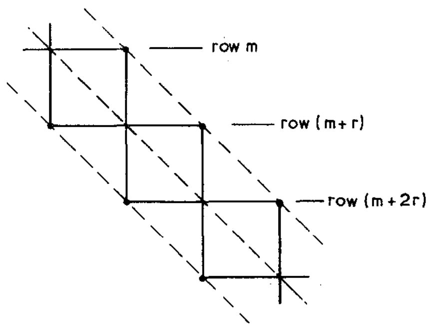

Fig. 2 Typical location of matrix coefficients determining the bandwidth at termination of the reduction.

The true minimum bandwidth is known to be 9 when a suitable sequence would be 1, 5, 9, 2, 6, 10, 3, 7, 11, 4, 8, 12.

Typical of this situation, for larger meshes, is the location of coefficients, which determine the bandwidth, on a square lattice as shown in Fig. 2. When rows m, $m + r$, $m + 2r$, etc. determine the bandwidth it is evident that a single interchange involving any of these rows will increase the bandwidth. For a similar mesh to the example but of size 5 by 10 instead of 2 by 3 and labeled in a similar manner, one such lattice would be at $m = 1$, $r = 12$.

Monotonic reduction of the bandwidth may be achieved by a series of simultaneous multiple cyclic interchanges involving many rows and columns. The procedure for finding cyclic interchanges is lengthy, and two alternatives are proposed.

Convergence either to the correct or pseudo minimum is dependent on the initial sequence. Thus, by starting each time with a random sequence and performing several reduction runs using the single interchange technique, if sufficient runs are performed one could give the true minimum. This proves simple but is unreliable.

A more rational alternative is to consider all possible sequences, but to discard blocks of sequences which are invalid at an early stage. Such an algorithm is described by Alway and Martin,[2] who also give an insight into the constraints imposed by the location of the coefficients which lead to the failure of the single interchange technique.

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Reply by Author to J. Barlow and C. G. Marples

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The conception of a theoretically failure-proof algorithm for the bandwidth minimization of the stiffness matrices would be a very simple task for a programmer. In fact, if n is the number of nodes in a given configuration, the survey of the $n!$ possible permutations for the node labels would yield the absolute minimum bandwidth. But, another survey of the number expressed by $n!$ shows that this approach is practically bound to a complete failure with increasing n within the ranges of the actual problem sizes; i.e., for $n = 15$, $n! \approx 10^{12}$, and the computation time would be of the order of 10^3 hr for such problems in the actual computers. Is it possible to devise a general algorithm, which, starting from any given configuration, would follow a reasonably smooth path to the absolute minimum bandwidth? After a considerable amount of computations with practical problems and test cases, using various criteria, the authors' conclusion is that the answer to this question might be affirmative, but yet the results might not have any practical value if the computer time increases drastically. Therefore, whether or not an algorithm might yield an absolute minimum bandwidth is not important. The criterion for success of an algorithm depends on its efficiency of obtaining some appreciable reduction of bandwidth within a reasonable computer time. Therefore, in any article on this topic, reference to the computer time must be the most essential factor. Reference 1, which was brought to our attention through Ref. 2, does indeed reflect these difficulties without any computer time data for basis of comparison.

With reference to Fig. 1a and 1b of Ref. 2, the inspection of the successive rows and columns shows that the algorithm of Ref. 3 fails without single interchange. This failure is due to the special form of the connectivity matrix for 3 × 2 division of the rectangular mesh, but it is hard to conceive any other mesh configuration of the plate or other topological form for which the scheme would fail. Actually, a suitable disturbance, i.e., isolation of a point by eliminating its connection with all adjacent points or an interchange of the labels of two arbitrary points, will restart the relabeling procedure leading generally to an optimum solution. An algorithm for automatic disturbance had been tried during the development of the actual program but has been discarded because of the increasing computer time. Usually the physical nature of the problem, i.e., boundary conditions in structural analysis, might introduce this type of disturbance.

The reasoning in paragraph 2 of Ref. 1 does not shadow the success of our basic algorithm. The inspections of the statements in the program list of ARAN[3] from 1280–1460 will show that there is a provision for temporary increase of bandwidth. The conditional statement following statement 1105 does not permit utilization of this provision because the failure of the program had not been detected in the test cases until the preparation date of Ref. 3. The program, as implemented in Ref. 4 late in 1967, utilizes this provision. The simple modification is shown in the first line of the errata list. To prove this point and to show the efficiency of the program, a series of test cases for rectangular plates of various mesh divisions, similar to that shown in Fig. 1a of Ref. 2, have been

Table 1 Results of test cases with various mesh divisions for rectangular plate

Mesh division	Number of nodes	Computation time t	Reduction of band area at time t, %	Reduction of band area to absolute minimum, %
10 × 5	66	10 sec[a]	30	35
20 × 10	231	100 sec	−10[b]	
		200 sec	13	
		234 sec[a]	31.5	39
34 × 14	525	100 sec	−4	
		5 min	−13	
		10 min	−13	
		15 min	−8	
		20 min	0.6	
		25 min	20.6	
		29 min[a]	37.8	46.8

[a] The last numbers in the time lists are the times at which the program stopped.
[b] The negative percentages refer to an increase of bandwidth.

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